

# ON THE REAL-ANALYTIC INFINITESIMAL CR AUTOMORPHISM OF HYPERSURFACES OF INFINITE TYPE

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**ABSTRACT.** We consider a real smooth hypersurface  $M \subset \mathbb{C}^2$ , which is of D'Angelo infinite type at  $p \in M$ . The purpose of this paper is to show that the real vector space of tangential holomorphic vector field germs at  $p$  vanishing at  $p$  is either trivial or of real dimension 1.

## 1. INTRODUCTION

Let  $(M, p)$  be a real  $\mathcal{C}^1$ -smooth hypersurface germ at  $p \in \mathbb{C}^n$ . A smooth vector field germ  $(X, p)$  on  $M$  is called a *real-analytic infinitesimal CR automorphism germ at  $p$  of  $M$*  if there exists a holomorphic vector field germ  $(H, p)$  in  $\mathbb{C}^n$  such that  $H$  is tangent to  $M$ , i.e.  $\operatorname{Re} H$  is tangent to  $M$ , and  $X = \operatorname{Re} H|_M$ . We denote by  $\operatorname{hol}_0(M, p)$  the real vector space of holomorphic vector field germs  $(H, p)$  vanishing at  $p$  which are tangent to  $M$ .

For a real hypersurface in  $\mathbb{C}^n$ , the real-analytic infinitesimal CR automorphism is not easy to describe explicitly; besides, it is unknown in most cases. For instance, the study of  $\operatorname{hol}_0(M, p)$  of various hypersurfaces is given in [1, 3, 7, 10, 11]. However, these results are known for Levi nondegenerate hypersurfaces or more generally for Levi degenerate hypersurfaces of finite type. For various real  $\mathcal{C}^\infty$ -smooth hypersurfaces of D'Angelo infinite type in  $\mathbb{C}^2$ , explicit descriptions of  $\operatorname{hol}_0(M, p)$  are given in [2, 8, 9].

In this paper we shall prove that  $\operatorname{hol}_0(M, p)$  of a certain hypersurface of D'Angelo infinite type in  $\mathbb{C}^2$  is either trivial or of real dimension 1. To state the result explicitly, we need some notations and a definition. Taking the risk of confusion we employ the notations

$$P'(z) = P_z(z) = \frac{\partial P}{\partial z}(z), \quad f_z(z, t) = \frac{\partial f}{\partial z}(z, t), \quad f_t(z, t) = \frac{\partial f}{\partial t}(z, t)$$

throughout the article. Also denote by  $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$  for  $r > 0$  and by  $\Delta = \Delta_1$ . A function  $f$  defined on  $\Delta_r$  ( $r > 0$ ) is called to be *flat* at the origin if  $f(z) = o(|z|^n)$  for each  $n \in \mathbb{N}$  (cf. Definition 1).

The aim of this paper is to prove the following theorem.

**Theorem 1.** *Let  $(M, 0)$  be a real  $\mathcal{C}^1$ -smooth hypersurface germ at 0 defined by the equation  $\rho(z) := \rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + \operatorname{Im} z_1 Q(z_2, \operatorname{Im} z_1) = 0$  satisfying the conditions:*

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- (1)  $P, Q$  are  $\mathcal{C}^1$ -smooth with  $P(0) = Q(0, 0) = 0$ ,
- (2)  $P(z_2) > 0$  for any  $z_2 \neq 0$ , and
- (3)  $P(z_2), P'(z_2)$  are flat at  $z_2 = 0$ .

Then  $\dim_{\mathbb{R}} \text{hol}_0(M, p) \leq 1$ .

*Remark 1.* When  $P, Q$  are  $\mathcal{C}^\infty$ -smooth, the condition (3) simply says that  $P$  vanishes to infinite order at 0 and moreover 0 is a point of D'Angelo infinite type.

In the case  $M$  is *radially symmetric* in  $z_2$ , i.e.  $P(z_2) = P(|z_2|)$  and  $Q(z_2, t) = Q(|z_2|, t)$  for any  $z_2$  and  $t$ , it is well-known that  $iz_2 \frac{\partial}{\partial z_2}$  is tangent to  $M$  (see cf. [2]). Therefore, by Theorem 1 one gets the following corollary, which is a slight generalization of the main result in [2].

**Corollary 1.** *Let  $(M, 0)$  be a real  $\mathcal{C}^1$ -smooth hypersurface germ at 0 defined by the equation  $\rho(z) := \rho(z_1, z_2) = \text{Re } z_1 + P(z_2) + \text{Im } z_1 Q(z_2, \text{Im } z_1) = 0$  satisfying the conditions:*

- (1)  $P, Q$  are  $\mathcal{C}^1$ -smooth with  $P(0) = Q(0, 0) = 0$ ,
- (2)  $P(z_2) = P(|z_2|)$ ,  $Q(z_2, t) = Q(|z_2|, t)$  for any  $z_2$  and  $t$ ,
- (3)  $P(z_2) > 0$  for any  $z_2 \neq 0$ , and
- (4)  $P(z_2), P'(z_2)$  are flat at  $z_2 = 0$ .

Then  $\text{hol}_0(M, 0) = \{i\beta z_2 \frac{\partial}{\partial z_2} : \beta \in \mathbb{R}\}$ .

Next, we shall give an explicit description for real-analytic infinitesimal CR automorphisms of another class of real hypersurfaces in  $\mathbb{C}^2$ .

Let  $a(z) = \sum_{n=1}^{\infty} a_n z^n$  be a nonzero holomorphic function defined on  $\Delta_{\epsilon_0}$  ( $\epsilon_0 > 0$ ) and let  $p, q$  be  $\mathcal{C}^1$ -smooth functions defined respectively on  $(0, \epsilon_0)$  and  $[0, \epsilon_0]$  satisfying that  $q(0) = 0$  and that  $g(z), g'(z)$  are flat at 0, where  $g$  is a  $\mathcal{C}^1$ -smooth function given by

$$g(z) = \begin{cases} e^{p(|z|)} & \text{if } 0 < |z| < \epsilon_0 \\ 0 & \text{if } z = 0. \end{cases}$$

Denote by  $M(a, \alpha, p, q)$  the germ at  $(0, 0)$  of a real hypersurface defined by

$$\rho(z_1, z_2) := \text{Re } z_1 + P(z_2) + f(z_2, \text{Im } z_1) = 0,$$

where  $f$  and  $P$  are respectively defined on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$  ( $\delta_0 > 0$  small enough) and  $\Delta_{\epsilon_0}$  by

$$f(z_2, t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos(R(z_2) + \alpha t)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0, \end{cases}$$

where  $R(z_2) = q(|z_2|) - \text{Re} \left( \sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right)$  for all  $z_2 \in \Delta_{\epsilon_0}$ , and

$$P(z_2) = \begin{cases} \frac{1}{\alpha} \log [1 + \alpha P_1(z_2)] & \text{if } \alpha \neq 0 \\ P_1(z_2) & \text{if } \alpha = 0, \end{cases}$$

where

$$P_1(z_2) = \exp \left[ p(|z_2|) + \text{Re} \left( \sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log |\cos(R(z_2))| \right]$$

for all  $z_2 \in \Delta_{\epsilon_0}^*$  and  $P_1(0) = 0$ .

It is easily checked that  $M(a, \alpha, p, q)$  is  $\mathcal{C}^1$ -smooth and moreover  $P(z_2), P'(z_2)$  are flat at 0. Furthermore, we note that  $q, p$  can be chosen, e.g.  $q(t) = 0$  and

$p(t) = -\frac{1}{t^\alpha}$  ( $\alpha > 0$ ) for all  $t > 0$ , so that  $P, R$  are  $\mathcal{C}^\infty$ -smooth in  $\Delta_{\epsilon_0}$  and  $P$  is flat at 0, and hence  $M(a, \alpha, p, q)$  is  $\mathcal{C}^\infty$ -smooth and of D'Angelo infinite type.

It follows from Theorem 3 in Appendix that the holomorphic vector field

$$H^{a,\alpha}(z_1, z_2) := L^\alpha(z_1)a(z_2)\frac{\partial}{\partial z_1} + iz_2\frac{\partial}{\partial z_2},$$

where

$$L^\alpha(z_1) = \begin{cases} \frac{1}{\alpha}(\exp(\alpha z_1) - 1) & \text{if } \alpha \neq 0 \\ z_1 & \text{if } \alpha = 0, \end{cases}$$

is tangent to  $M(a, \alpha, p, q)$ . Hence, by Theorem 1 we obtain following corollary.

**Corollary 2.**  $\text{hol}_0(M(a, \alpha, p, q), 0) = \{\beta H^{a,\alpha} : \beta \in \mathbb{R}\}$ .

This paper is organized as follows. In Section 2, we recall several definitions and give several technical lemmas. Next, the proof of Theorem 1 is given in Section 3. Finally, a theorem is pointed out in Appendix.

## 2. PRELIMINARIES

In this section, we shall recall several definitions and introduce two technical lemmas used in the proof of Theorem 1. In what follows,  $\lesssim$  and  $\gtrsim$  denote inequalities up to a positive constant. In addition, we use  $\approx$  for the combination of  $\lesssim$  and  $\gtrsim$ .

**Definition 1.** A function  $f : \Delta_{\epsilon_0} \rightarrow \mathbb{C}$  ( $\epsilon_0 > 0$ ) is called to be *flat* at  $z = 0$  if for each  $n \in \mathbb{N}$  there exist positive constants  $C, \epsilon > 0$ , depending only on  $n$ , with  $0 < \epsilon < \epsilon_0$  such that

$$|f(z)| \leq C|z|^n$$

for all  $z \in \Delta_\epsilon$ .

We note that in the above definition we do not need the smoothness of the function  $f$ . For example, the following function

$$f(z) = \begin{cases} \frac{1}{n}e^{-\frac{1}{|z|^2}} & \text{if } \frac{1}{n+1} < |z| \leq \frac{1}{n}, n = 1, 2, \dots \\ 0 & \text{if } z = 0 \end{cases}$$

is flat at  $z = 0$  but not continuous on  $\Delta$ . However, if  $f \in \mathcal{C}^\infty(\Delta_{\epsilon_0})$  then it follows from the Taylor's theorem that  $f$  is flat at  $z = 0$  if and only if

$$\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} f(0) = 0$$

for every  $m, n \in \mathbb{N}$ , i.e.,  $f$  vanishes to infinite order at 0. Consequently, if  $f \in \mathcal{C}^\infty(\Delta_{\epsilon_0})$  is flat at 0 then  $\frac{\partial^{m+n} f}{\partial z^m \partial \bar{z}^n}$  is also flat at 0 for each  $m, n \in \mathbb{N}$ .

Let  $F$  be a  $\mathcal{C}^1$ -smooth complex-valued function defined in a neighborhood  $U$  of the origin in the complex plane. We consider the autonomous dynamical system

$$\frac{dz}{dt} = F(z), \quad z(0) = z_0 \in U. \quad (1)$$

First of all, let us recall several definitions.

**Definition 2.** A state  $\hat{z} \in U$  is called an equilibrium of (1) if  $F(\hat{z}) = 0$ .

**Definition 3.** An equilibrium,  $\hat{z}$ , of (1) is called locally asymptotically stable if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|z_0 - \hat{z}| < \delta$  implies that  $|z(t) - \hat{z}| < \epsilon$  for all  $t \geq 0$  and  $\lim_{t \rightarrow +\infty} z(t) = 0$ .

The following lemma is a generalization of Lemma 3 in [8] and plays a key role in the proof of Theorem 1.

**Lemma 1.** Let  $P : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$  be a  $C^1$ -smooth function satisfying that  $P(z) > 0$  for any  $z \in \Delta_{\epsilon_0}^*$  and that  $P$  is flat at 0. If  $a, b$  are complex numbers and if  $g_0, g_1, g_2$  are  $C^1$ -smooth functions defined on  $\Delta_{\epsilon_0}$  satisfying:

(A1)  $g_0(z) = O(|z|)$ ,  $g_1(z) = O(|z|^\ell)$ , and  $g_2(z) = o(|z|^m)$ , and

(A2)  $\operatorname{Re} \left[ az^m + \frac{1}{P^n(z)} \left( bz^\ell (1 + g_0(z)) \frac{P'(z)}{P(z)} + g_1(z) \right) \right] = g_2(z)$  for every  $z \in \Delta_{\epsilon_0}^*$

for any nonnegative integers  $\ell, m$  and  $n$  except for the following two cases

(E1)  $\ell = 1$  and  $\operatorname{Re} b = 0$ , and

(E2)  $m = 0$  and  $\operatorname{Re} a = 0$

then  $ab = 0$ .

*Proof.* We shall prove the lemma by contradiction. Suppose that there exist non-zero complex numbers  $a, b \in \mathbb{C}^*$  such that the identity in (A2) holds with the smooth functions  $g_0, g_1$ , and  $g_2$  satisfying the growth conditions specified in (A1).

Denote by  $F(z) := \frac{1}{2} \log P(z)$  for all  $z \in \Delta_{\epsilon_0}^*$  and by  $f(z) := bz^\ell(1 + g_0(z))$  for all  $z \in \Delta_{\epsilon_0}$ .

**Case 1.  $\ell = 0$ :**

Let  $\gamma : [0, \delta_0) \rightarrow \Delta_{\epsilon_0}$  ( $\delta_0 > 0$ ) be the solution of the initial-value problem

$$\frac{d\gamma(t)}{dt} = b + bg_0(\gamma(t)), \quad \gamma(0) = 0.$$

Let us denote by  $u(t) := F(\gamma(t))$ ,  $0 < t < \delta_0$ . By (A2), it follows that  $u'(t)$  is bounded on the interval  $(0, \delta_0)$ . Integration shows that  $u(t)$  is also bounded on  $(0, \delta_0)$ . But this is impossible since  $u(t) \rightarrow -\infty$  as  $t \downarrow 0$ .

**Case 2.  $\ell = 1$ :**

By (E1), we have  $b_1 := \operatorname{Re}(b) \neq 0$ . Assume momentarily that  $b_1 < 0$ . Let  $\gamma : [t_0, +\infty) \rightarrow \Delta_{\epsilon_0}^*$  ( $t_0 > 0$ ) be the solution of the initial-value problem

$$\frac{d\gamma(t)}{dt} = b\gamma(t) \left( 1 + g_0(\gamma(t)) \right), \quad \gamma(t_0) = z_0 \in \Delta_{\epsilon_0}^*. \quad (2)$$

Thanks to [4, Theorem 5], the system (2) is locally trajectory equivalent at the origin to the system

$$\frac{dz}{dt} = bz(t).$$

It is well-known that the origin is a locally asymptotically stable equilibrium of the above differential equation. Therefore, we have  $\gamma(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . Moreover, we can assume that  $|\gamma(t)| < r_1$  for every  $t_0 < t < +\infty$ , where  $r_1 := 1/2$  if  $\operatorname{Im}(b) = 0$  and  $r_1 := \min\{1/2, |b_1|/(4|\operatorname{Im}(b)|)\}$  if otherwise. This implies that  $\operatorname{Re} \left( b(1 + g_0(\gamma(t))) \right) < b_1/4 < 0$ . Integration and a simple estimation tell us that

$$|\gamma(t)| \leq |\gamma(t_0)| \exp \left( b_1 t / 4 \right), \quad \forall t > t_0.$$

Consequently, this in turn yields that  $t \lesssim \log \frac{1}{|\gamma(t)|}$ .

Denote by  $u(t) := F(\gamma(t))$  for  $t \geq t_0$ . Then, it follows from (A2) that  $u'(t)$  is bounded on  $(t_0, +\infty)$ , and thus  $|u(t)| \lesssim t$ . Therefore, there exists a constant  $A > 0$  such that  $|u(t)| \leq A \log \frac{1}{|\gamma(t)|}$  for all  $t > t_0$ . Hence we obtain, for all  $t > t_0$ , that  $\log P(\gamma(t)) = 2u(t) \geq -2A \log \frac{1}{|\gamma(t)|}$ , and thus

$$P(\gamma(t)) \geq |\gamma(t)|^{2A}, \quad t \geq t_0.$$

Hence we arrive at

$$\lim_{t \rightarrow +\infty} \frac{P(\gamma(t))}{|\gamma(t)|^{2A+1}} = +\infty,$$

which is impossible since  $P$  is flat at 0. The case  $b_1 > 0$  is similar, with considering the side  $t < 0$  instead.

**Case 3.  $\ell = k + 1 \geq 2$ :**

Let  $\gamma : [t_0, +\infty) \rightarrow \Delta_{\epsilon_0}^*$  ( $t_0 > 0$ ) be a solution of the initial-value problem

$$\frac{d\gamma(t)}{dt} = f(\gamma(t)) = b\gamma^{k+1}(t) \left(1 + g_0(\gamma(t))\right), \quad \gamma(t_0) = z_0 \in \Delta_{\epsilon_0}^*. \quad (3)$$

According to [4, Theorem 5], the system (3) is locally trajectory equivalent at the origin to the system

$$\frac{dz}{dt} = bz^{k+1}(t).$$

Hence, it follows from [12, Theorem 1] that  $\gamma(t) \rightarrow 0$  as  $t \rightarrow +\infty$ .

Now we shall estimate  $\gamma(t)$ . Indeed, integration shows that

$$\frac{1}{\gamma^k(t)} = c - kbt(1 + \epsilon(t)), \quad \forall t > t_0, \quad (4)$$

where  $c$  is a constant depending only on the initial condition and

$$\epsilon(t) = \frac{\int_{t_0}^t g_0(\gamma(s))ds}{t - t_0} \quad \text{for every } t > t_0.$$

Choose  $\delta > 0$  such that either  $\arg(b(1+z)) \in (0, 2\pi)$  for all  $z \in \Delta_\delta$  (for the case  $\text{Im}(b) \neq 0$ ) or  $\arg(b(1+z)) \in (-\pi/2, 3\pi/2)$  for all  $z \in \Delta_\delta$  (for the case  $\text{Im}(b) = 0$ ). Moreover, without loss of generality we can assume that  $|g_0(\gamma(t))| < \delta$  for all  $t > t_0$  and hence  $|\epsilon(t)| < \delta$  for all  $t > t_0$ . Therefore, by changing the initial condition  $\gamma(t_0) = z_0$  if necessary, we may assume that either  $c - kbt(1 + \epsilon(t)), c - kbt \in \mathbb{C} \setminus [0, +\infty)$  for all  $t \in [t_0, +\infty)$  or  $c - kbt(1 + \epsilon(t)), c - kbt \in \mathbb{C} \setminus (-\infty i, 0]$  for all  $t \in [t_0, +\infty)$ . Without loss of generality, we can assume that the first case occurs.

Notice that  $\omega_j(t) := \tau^{-j} \sqrt[k]{c - kbt}$ ,  $j = 0, \dots, k-1$ , are solutions of the equation

$$\frac{dz}{dt} = bz^{k+1},$$

where  $\tau := e^{i2\pi/k}$ . Furthermore, for each  $j \in \{0, 1, \dots, k-1\}$  let  $\theta_j(t)$  ( $t \geq t_0$ ) be the solution of the equation

$$\theta_j'(t) = f(\omega_j(t) + \theta_j(t)) - b\omega_j^{k+1}(t)$$

satisfying  $\theta_j(t_0) = 0$ . Then  $\gamma_j(t) := \omega_j(t) + \theta_j(t)$  ( $t > t_0$ ),  $j = 0, 1, \dots, k-1$ , are solutions of

$$\frac{dz}{dt} = f(z).$$

Moreover, again by changing the initial condition  $\gamma(t_0) = z_0$  if necessary we can assume that  $|g_0(\gamma_j(t))| < \delta$  for every  $j = 0, 1, \dots, k-1$  and for every  $t > t_0$ . In addition, integration shows that

$$\frac{1}{\gamma_j^k(t)} = c - kbt(1 + \epsilon_j(t)), \quad \forall t > t_0, \quad (5)$$

where  $\epsilon_j(t) = \frac{\int_{t_0}^t g_o(\gamma_j(s))ds}{t-t_0}$  for every  $t > t_0$  and for every  $j = 0, 1, \dots, k-1$ . Hence, we obtain the following.

$$\begin{aligned} \gamma_j(t) &= \tau^{-j} \sqrt[k]{c - kbt(1 + \epsilon_j(t))} \\ &= \tau^{-j} \sqrt[k]{|c - kbt(1 + \epsilon(t))|} e^{-i \arg(c - kbt(1 + \epsilon(t))) / k} \\ &= \sqrt[k]{|c - kbt(1 + \epsilon(t))|} e^{-i \arg(c - kbt(1 + \epsilon(t))) / k - i 2\pi j / k}, \end{aligned}$$

where  $0 < \arg(c - kbt(1 + \epsilon_j(t))) / k < 2\pi$ , for every  $j = 0, 1, \dots, k-1$ . Consequently,

$$|\gamma_j(t)| \approx \frac{1}{|t|^{1/k}} \text{ for all } t \geq t_0 \text{ and for all } j = 0, 1, \dots, k-1.$$

Let  $u_j(t) := F(\gamma_j(t))$  for  $j = 0, 1, \dots, k-1$ . It follows from (A2) that

$$u_j'(t) = -P^n(\gamma_j(t)) \left( \operatorname{Re}(a\gamma_j^m(t) + o(|\gamma_j(t)|^m)) \right) + O(|\gamma_j(t)|^{k+1}) \quad (6)$$

for all  $t > t_0$  and for all  $j = 0, 1, \dots, k-1$ .

We now consider the following.

Subcase 3.1:  $n \geq 1$ .

Since  $P$  is flat at the origin, (6) and the discussion above imply

$$\begin{aligned} |u_0'(t)| &\lesssim P^n(\gamma_0(t)) |\gamma_0(t)|^m + \frac{1}{t^{1+1/k}} \\ &\lesssim P^n(\gamma_0(t)) + \frac{1}{t^{1+1/k}} \\ &\lesssim \frac{P^n(\gamma_0(t))}{|\gamma_0(t)|^{2k}} \frac{1}{t^2} + \frac{1}{t^{1+1/k}} \\ &\lesssim \frac{1}{t^2} + \frac{1}{t^{1+1/k}} \\ &\lesssim \frac{1}{t^{1+1/k}} \end{aligned}$$

for all  $t \geq t_0$ . This in turn yields

$$\begin{aligned} |u_0(t)| &\lesssim |u_0(t_0)| + \int_{t_0}^t \frac{1}{s^{1+1/k}} ds \\ &\lesssim |u_0(t_0)| + k \left( \frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}} \right) \\ &\lesssim 1 \end{aligned}$$

for all  $t > t_0$ . This is a contradiction, because  $\lim_{t \rightarrow \infty} u_0(t) = -\infty$ .

Subcase 3.2:  $n = 0$ .

We again divide the argument into 4 sub-subcases.

Subcase 3.2.1:  $m/k > 1$ .

It follows from (6) that

$$|u'_0(t)| \lesssim \frac{1}{t^{m/k}} + \frac{1}{t^{1+1/k}}$$

for all  $t \geq t_0$ . Hence, we get

$$\begin{aligned} |u_0(t)| &\lesssim |u_0(t_0)| + \int_{t_0}^t \left( \frac{1}{s^{m/k}} + \frac{1}{s^{1+1/k}} \right) ds \\ &\lesssim |u_0(t_0)| + \frac{k}{m-k} \left( \frac{1}{t_0^{m/k-1}} - \frac{1}{t^{m/k-1}} \right) + k \left( \frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}} \right) \\ &\lesssim 1 \end{aligned}$$

for all  $t > t_0$ , which contradicts  $\lim_{t \rightarrow +\infty} u_0(t) = -\infty$ .

Subcase 3.2.2:  $m/k = 1$ .

Here, (6) again implies

$$|u'_0(t)| \lesssim \frac{1}{t} + \frac{1}{t^{1+1/k}} \lesssim \frac{1}{t}$$

for all  $t \geq t_0$ . Consequently,

$$\begin{aligned} |u_0(t)| &\lesssim |u_0(t_0)| + \int_{t_0}^t \frac{1}{s} ds \\ &\lesssim |u_0(t_0)| + (\log t - \log t_0) \\ &\lesssim \log t \\ &\lesssim \log \frac{1}{|\gamma_0(t)|} \end{aligned}$$

for all  $t > t_0$ . Therefore there exists a constant  $A > 0$  such that  $|u_0(t)| \leq A \log \frac{1}{|\gamma_0(t)|}$  for all  $t > t_0$ . Hence for all  $t > t_0$ ,  $\log P(\gamma_0(t)) = 2u(t) \geq -2A \log \frac{1}{|\gamma_0(t)|}$ , and thus

$$P(\gamma_0(t)) \geq |\gamma_0(t)|^{2A}, \quad \forall t \geq t_0.$$

This ensures

$$\lim_{t \rightarrow +\infty} \frac{P(\gamma_0(t))}{|\gamma_0(t)|^{2A+1}} = +\infty,$$

which is again impossible since  $P$  is flat at 0.

Subcase 3.2.3:  $m = 0$ .

Let  $h(t) := u_0(t) + \operatorname{Re}(a)t$ . Recall that in this case we have (E2) which says  $\operatorname{Re} a \neq 0$ . Assume momentarily that  $\operatorname{Re}(a) < 0$ . (The case that  $\operatorname{Re}(a) > 0$  will follow by a similar argument.)

By (6), there is a constant  $B > 0$  such that

$$|h'(t)| \leq \frac{1}{2} |\operatorname{Re}(a)| + B \frac{1}{t^{1+1/k}}.$$

Therefore,

$$\begin{aligned} |h(t)| &\leq |h(t_0)| + \frac{1}{2} |\operatorname{Re}(a)|(t - t_0) + B \int_{t_0}^t \frac{1}{s^{1+1/k}} ds \\ &\leq |h(t_0)| + \frac{1}{2} |\operatorname{Re}(a)|(t - t_0) + kB \left( \frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}} \right) \end{aligned}$$

for all  $t > t_0$ . Thus

$$\begin{aligned} u_0(t) &\geq -\operatorname{Re}(a)t - |h(t)| \\ &\geq |\operatorname{Re}(a)|t - |h(t_0)| - \frac{1}{2}|\operatorname{Re}(a)|(t - t_0) - kB\left(\frac{1}{t_0^{1/k}} - \frac{1}{t^{1/k}}\right) \\ &\gtrsim t \end{aligned}$$

for all  $t > t_0$ . It means that  $u_0(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ , and it is hence absurd.

Subcase 3.2.4:  $0 < \frac{m}{k} < 1$ . Assume for a moment that  $m$  and  $k$  are relatively prime. (In the end, it will become obvious that this assumption can be taken without loss of generality.) Then  $\tau^m$  is a primitive  $k$ -th root of unity. Therefore there exist  $j_0, j_1 \in \{1, \dots, k-1\}$  such that  $\pi/2 < \arg(\tau^{mj_0}) \leq \pi$  and  $-\pi \leq \arg(\tau^{mj_1}) < -\pi/2$ . Hence, it follows that there exists  $j \in \{0, \dots, k-1\}$  such that  $\cos(\arg(a/b) + \frac{k-m}{k}\arg(-b) - 2\pi mj/k) > 0$ . Denote by

$$A := \frac{|a|}{(k-m)|b|} \cos\left(\arg(a/b) + \frac{k-m}{k}\arg(-b) - 2\pi mj/k\right) > 0,$$

a positive constant. Now let

$$h_j(t) := u_j(t) + \operatorname{Re}(\tau^{-mj} \frac{a}{-b(k-m)} (c - kbt)^{1-m/k}).$$

Note that  $\arg(c - kbt) \rightarrow \arg(-b)$  as  $t \rightarrow +\infty$  and  $\delta > 0$  can be chosen so small that there exists  $t_1 > t_0$  big enough such that

$$\begin{aligned} \left| \gamma_j^m(t) - \tau^{-mj} \left( \frac{1}{c - kbt} \right)^{m/k} \right| &= \left| \frac{1}{(c - kb(t + \epsilon_j(t)))^{m/k}} \left[ 1 - \left( 1 - \frac{kbt\epsilon_j(t)}{c - kbt} \right)^{m/k} \right] \right| \\ &\leq \frac{k-m}{8k|a|} A(k|b|)^{1-m/k} \frac{1}{t^{m/k}} \end{aligned}$$

for every  $t > t_1$ . Hence it follows from (6) that there exist positive constants  $B$  and  $t_2$  ( $t_2 > t_1$ ) such that

$$|h'_j(t)| \leq \frac{k-m}{4k} A(k|b|)^{1-m/k} \frac{1}{t^{m/k}} + \frac{B}{t^{1+1/k}}$$

and

$$\begin{aligned} \cos\left(\arg(a/b) + \frac{k-m}{k}\arg(c - kbt) - 2mj\pi/k\right) \\ \geq \frac{1}{2} \cos\left(\arg(a/b) + \frac{k-m}{k}\arg(-b) - 2mj\pi/k\right) \end{aligned}$$

for every  $t \geq t_2$ . Thus we have

$$\begin{aligned} |h_j(t)| &\leq |h_j(t_2)| + A(k|b|)^{1-m/k} \frac{k-m}{4k} \int_{t_2}^t s^{-m/k} ds + B \int_{t_2}^t s^{-1-1/k} ds \\ &\leq |h_j(t_2)| + \frac{A}{4} (k|b|)^{1-m/k} (t^{1-m/k} - t_2^{1-m/k}) + kB(t_2^{-1/k} - t^{-1/k}) \end{aligned}$$



for  $t > t_2$ . Hence

$$\begin{aligned}
u_j(t) &\geq -\operatorname{Re}\left(\frac{a\tau^{-mj}}{-kb(1-m/k)}(c-kbt)^{1-m/k}\right) - |h_j(t)| \\
&\geq \frac{|a|}{|b|(k-m)}|c-kbt|^{1-m/k} \cos\left(\arg(a/b) + \frac{(k-m)\arg(c-kbt) - 2mj\pi}{k}\right) - |h_j(t_2)| \\
&\quad - \frac{A}{4}(k|b|)^{1-m/k}(t^{1-m/k} - t_2^{1-m/k}) - kB(t_2^{-1/k} - t^{-1/k}) \\
&\geq \frac{A}{2}|c-kbt|^{1-m/k} - |h_j(t_2)| \\
&\quad - \frac{A}{4}(k|b|)^{1-m/k}(t^{1-m/k} - t_2^{1-m/k}) - kB(t_2^{-1/k} - t^{-1/k}) \\
&\gtrsim t^{1-m/k}
\end{aligned}$$

for  $t > t_2$ . This implies that  $u_j(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ , which is absurd since  $\log P(z) \rightarrow -\infty$  as  $z \rightarrow 0$ .

Hence all the cases are covered, and the proof of Lemma 1 is finally complete.  $\square$

Following the proof of Lemma 1, we have the following lemma.

**Lemma 2.** *Let  $P : \Delta_{\epsilon_0} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$ -smooth function satisfying that  $P(z) > 0$  for any  $z \in \Delta_{\epsilon_0}^*$  and that  $P$  is flat at 0. If  $b$  is a complex number and if  $g$  is a  $\mathcal{C}^1$ -smooth function defined on  $\Delta_{\epsilon_0}$  satisfying:*

- (B1)  $g(z) = O(|z|^{k+1})$ , and
- (B2)  $\operatorname{Re}\left[(bz^k + g(z))P'(z)\right] = 0$  for every  $z \in \Delta_{\epsilon_0}$

*for some nonnegative integer  $k$ , except the case  $k = 1$  and  $\operatorname{Re}(b) = 0$ , then  $b = 0$ .*

### 3. THE VECTOR SPACE OF TANGENTIAL HOLOMORPHIC VECTOR FIELDS

This section is devoted to the proof of Theorem 1. First of all, we need the following theorem.

**Theorem 2.** *If a holomorphic vector field germ  $(H, 0)$  vanishing at the origin which contains no nonzero term  $i\beta z_2 \frac{\partial}{\partial z_2}$  ( $\beta \in \mathbb{R}^*$ ) and is tangent to a real  $\mathcal{C}^1$ -smooth hypersurface germ  $(M, 0)$  defined by the equation  $\rho(z) := \rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + \operatorname{Im} z_1 Q(z_2, \operatorname{Im} z_1) = 0$  satisfying the conditions:*

- (1)  $P, Q$  are  $\mathcal{C}^1$ -smooth with  $P(0) = Q(0, 0) = 0$ ,
- (2)  $P(z_2) > 0$  for any  $z_2 \neq 0$ , and
- (3)  $P(z_2), P'(z_2)$  are flat at  $z_2 = 0$ ,

*then  $H = 0$ .*

*Proof.* The CR hypersurface germ  $(M, 0)$  at the origin in  $\mathbb{C}^2$  under consideration is defined by the equation  $\rho(z_1, z_2) = 0$ , where

$$\rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + (\operatorname{Im} z_1) Q(z_2, \operatorname{Im} z_1) = 0,$$

where  $P, Q$  are  $\mathcal{C}^1$ -smooth functions satisfying the three conditions specified in the hypothesis of our lemma. Recall that  $P(z_2), P'(z_2)$  are flat at  $z_2 = 0$  in particular.

Then we consider a holomorphic vector field  $H = h_1(z_1, z_2) \frac{\partial}{\partial z_1} + h_2(z_1, z_2) \frac{\partial}{\partial z_2}$  defined on a neighborhood of the origin satisfying that  $H(0) = 0$  and that  $H$  contains no nonzero term  $i\beta z_2 \frac{\partial}{\partial z_2}$  ( $\beta \in \mathbb{R}^*$ ). We only consider  $H$  that is tangent to  $M$ , which means that they satisfy the identity

$$(\operatorname{Re} H)\rho(z) = 0, \forall z \in M. \quad (7)$$

Expand  $h_1$  and  $h_2$  into the Taylor series at the origin so that

$$h_1(z_1, z_2) = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k \text{ and } h_2(z_1, z_2) = \sum_{j,k=0}^{\infty} b_{jk} z_1^j z_2^k,$$

where  $a_{jk}, b_{jk} \in \mathbb{C}$ . We note that  $a_{00} = b_{00} = 0$  since  $h_1(0, 0) = h_2(0, 0) = 0$ .

By a simple computation, we have

$$\begin{aligned} \rho_{z_1}(z_1, z_2) &= \frac{1}{2} + \frac{Q(z_2, \operatorname{Im} z_1)}{2i} + (\operatorname{Im} z_1)Q_{z_1}(z_2, \operatorname{Im} z_1), \\ \rho_{z_2}(z_1, z_2) &= P'(z_2) + (\operatorname{Im} z_1)Q_{z_2}(z_2, \operatorname{Im} z_1), \end{aligned}$$

and the equation (7) can thus be re-written as

$$\begin{aligned} \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{Q(z_2, \operatorname{Im} z_1)}{2i} + (\operatorname{Im} z_1)Q_{z_1}(z_2, \operatorname{Im} z_1) \right) h_1(z_1, z_2) \right. \\ \left. + \left( P'(z_2) + (\operatorname{Im} z_1)Q_{z_2}(z_2, \operatorname{Im} z_1) \right) h_2(z_1, z_2) \right] = 0 \end{aligned} \quad (8)$$

for all  $(z_1, z_2) \in M$ .

Since  $(it - P(z_2) - tQ(z_2, t), z_2) \in M$  for any  $t \in \mathbb{R}$  with  $t$  small enough, the above equation again admits a new form

$$\begin{aligned} \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{Q(z_2, t)}{2i} + tQ_{z_1}(z_2, t) \right) \sum_{j,k=0}^{\infty} a_{jk} (it - P(z_2) - tQ(z_2, t))^j z_2^k \right. \\ \left. + \left( P'(z_2) + tQ_{z_2}(z_2, t) \right) \sum_{m,n=0}^{\infty} b_{mn} (it - P(z_2) - tQ(z_2, t))^m z_2^n \right] = 0 \end{aligned} \quad (9)$$

for all  $z_2 \in \mathbb{C}$  and for all  $t \in \mathbb{R}$  with  $|z_2| < \epsilon_0$  and  $|t| < \delta_0$ , where  $\epsilon_0 > 0$  and  $\delta_0 > 0$  are small enough.

The goal is to show that  $H \equiv 0$ . Indeed, striving for a contradiction, suppose that  $H \not\equiv 0$ . We notice that if  $h_2 \equiv 0$  then (8) shows that  $h_1 \equiv 0$ . So, we must have  $h_2 \not\equiv 0$ .

We now divide the argument into two cases as follows.

**Case 1.  $h_1 \not\equiv 0$ .** In this case let us denote by  $j_0$  the smallest integer such that  $a_{j_0 k} \neq 0$  for some integer  $k$ . Then let  $k_0$  be the smallest integer such that  $a_{j_0 k_0} \neq 0$ . Similarly, let  $m_0$  be the smallest integer such that  $b_{m_0 n} \neq 0$  for some integer  $n$ . Then denote by  $n_0$  the smallest integer such that  $b_{m_0 n_0} \neq 0$ . We can see that  $j_0 \geq 1$  if  $k_0 = 0$  and  $m_0 \geq 1$  if  $n_0 = 0$ .

Since  $P(z_2) = o(|z_2|^j)$  for any  $j \in \mathbb{N}$ , inserting  $t = \alpha P(z_2)$  into (9), where  $\alpha \in \mathbb{R}$  will be chosen later, one has

$$\begin{aligned} \operatorname{Re} \left[ \frac{1}{2} a_{j_0 k_0} (i\alpha - 1)^{j_0} (P(z_2))^{j_0} (z_2^{k_0} + o(|z_2|^{k_0})) + b_{m_0 n_0} (i\alpha - 1)^{m_0} (z_2^{n_0} + o(|z_2|^{n_0})) \right. \\ \left. \times (P(z_2))^{m_0} \left( P'(z_2) + \alpha P(z_2) Q_{z_2}(z_2, \alpha P(z_2)) \right) \right] = 0 \end{aligned} \quad (10)$$

for all  $z_2 \in \Delta_{\epsilon_0}$ . We note that in the case  $k_0 = 0$  and  $\operatorname{Re}(a_{j_0 0}) = 0$ ,  $\alpha$  can be chosen in such a way that  $\operatorname{Re}((i\alpha - 1)^{j_0} a_{j_0 0}) \neq 0$ . Then (10) yields that  $j_0 > m_0$  by virtue of the fact that  $P'(z_2), P(z_2)$  are flat at  $z_2 = 0$ . Hence, we conclude from Lemma 1 that  $m_0 = 0, n_0 = 1$ , and  $b_{0,1} = i\beta z_2$  for some  $\beta \in \mathbb{R}^*$ . This is a contradiction with the assumption  $H$  contains no nonzero term  $i\beta z_2 \frac{\partial}{\partial z_2}$ .

**Case 2.  $h_1 \equiv 0$ .** Let  $m_0, n_0$  be as in the Case 1. Since  $P(z_2) = o(|z_2|^{n_0})$ , letting  $t = 0$  in (9) one obtains that

$$\operatorname{Re}\left[b_{m_0 n_0}(z_2^{n_0} + o(|z_2|^{n_0}))P'(z_2)\right] = 0 \quad (11)$$

for all  $z_2 \in \Delta_{\epsilon_0}$ . Therefore, Lemma 2 yields that  $m_0 = 0, n_0 = 1$ , and  $b_{0,1} = i\beta z_2$  for some  $\beta \in \mathbb{R}^*$ , which is again impossible.

Altogether, the proof of our theorem is complete.  $\square$

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* Let  $H_1, H_2 \in \operatorname{hol}_0(M, p)$  be arbitrary. Then by Theorem 2 we have that  $H_j$  contains term  $i\beta_j z_2 \frac{\partial}{\partial z_2}$  ( $j = 1, 2$ ) for some  $\beta_1, \beta_2 \in \mathbb{R}$ . Therefore,  $\beta_2 H_1 - \beta_1 H_2$  does not contain a term  $i\beta z_2 \frac{\partial}{\partial z_2}$ . Hence, Theorem 2 again yields that  $\beta_2 H_1 - \beta_1 H_2 = 0$ , which proves the theorem.  $\square$

#### 4. APPENDIX

We recall the following theorem that gives examples of holomorphic vector fields and real hypersurfaces which are tangent.

**Theorem 3** (see Theorem 3 in [9]). *Let  $\alpha \in \mathbb{R}$  and let  $a(z) = \sum_{n=1}^{\infty} a_n z^n$  be a non-zero holomorphic function defined on a neighborhood of  $0 \in \mathbb{C}$ , where  $a_n \in \mathbb{C}$  for all  $n \geq 1$ . Then there exist positive numbers  $\epsilon_0, \delta_0 > 0$  such that the holomorphic vector field*

$$H^{a,\alpha}(z_1, z_2) = L^\alpha(z_1)a(z_2)\frac{\partial}{\partial z_1} + iz_2\frac{\partial}{\partial z_2},$$

where

$$L^\alpha(z_1) = \begin{cases} \frac{1}{\alpha}(\exp(\alpha z_1) - 1) & \text{if } \alpha \neq 0 \\ z_1 & \text{if } \alpha = 0, \end{cases}$$

is tangent to the  $\mathcal{C}^1$ -smooth hypersurface  $M$  given by

$$M = \{(z_1, z_2) \in \Delta_{\delta_0} \times \Delta_{\epsilon_0} : \rho(z_1, z_2) := \operatorname{Re} z_1 + P(z_2) + f(z_2, \operatorname{Im} z_1) = 0\},$$

where  $f$  and  $P$  are respectively defined on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$  and  $\Delta_{\epsilon_0}$  by

$$f(z_2, t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos(R(z_2) + \alpha t)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0, \end{cases}$$

where  $R(z_2) = q(|z_2|) - \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n\right)$  for all  $z_2 \in \Delta_{\epsilon_0}$ , and

$$P(z_2) = \begin{cases} \frac{1}{\alpha} \log [1 + \alpha P_1(z_2)] & \text{if } \alpha \neq 0 \\ P_1(z_2) & \text{if } \alpha = 0, \end{cases}$$

where

$$P_1(z_2) = \exp \left[ p(|z_2|) + \operatorname{Re} \left( \sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log |\cos(R(z_2))| \right]$$

for all  $z_2 \in \Delta_{\epsilon_0}^*$  and  $P_1(0) = 0$ , and  $q, p$  are reasonable functions defined on  $[0, \epsilon_0)$  and  $(0, \epsilon_0)$  respectively with  $q(0) = 0$  so that  $P, R$  are  $\mathcal{C}^1$ -smooth in  $\Delta_{\epsilon_0}$ .

*Proof.* First of all, it is easy to show that there is  $\epsilon_0 > 0$  such that we can choose a function  $q$  so that the function  $R$  defined as in the theorem is  $\mathcal{C}^1$ -smooth and  $|R(z_2)| \leq 1$  on  $\Delta_{\epsilon_0}$ . Choose  $\delta_0 = \frac{1}{2|\alpha|}$  if  $\alpha \neq 0$  and  $\delta_0 = +\infty$  if otherwise. Then the function  $f(z_2, t)$  given in the theorem is  $\mathcal{C}^1$ -smooth on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ . Moreover,  $f(z_2, t)$  is real analytic in  $t$  and  $\frac{\partial^m f}{\partial t^m}$  is  $\mathcal{C}^1$ -smooth on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$  for each  $m \in \mathbb{N}$ .

Next, let  $P_1, P, R$  be functions defined as in the theorem and let  $Q_0(z_2) := \tan(R(z_2))$  for all  $z_2 \in \Delta_{\epsilon_0}$ . By a direct computation, we have the following equations.

- (i)  $\operatorname{Re} \left[ iz_2 Q_{0z_2}(z_2) + \frac{1}{2} (1 + Q_0^2(z_2)) ia(z_2) \right] \equiv 0;$
- (ii)  $\operatorname{Re} \left[ iz_2 P_{1z_2}(z_2) - \left( \frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) a(z_2) P_1(z_2) \right] \equiv 0;$
- (iii)  $\operatorname{Re} \left[ iz_2 P_{z_2}(z_2) + \frac{\exp(-\alpha P(z_2)) - 1}{\alpha} \left( \frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) a(z_2) \right] \equiv 0$  for  $\alpha \neq 0;$
- (iv)  $\left( i + f_t(z_2, t) \exp(\alpha(it - f(z_2, t))) \right) \equiv i + Q_0(z_2);$
- (v)  $\operatorname{Re} \left[ 2i\alpha z_2 f_{z_2}(z_2, t) + \left( f_t(z_2, t) - Q_0(z_2) \right) ia(z_2) \right] \equiv 0$

on  $\Delta_{\epsilon_0}$  for any  $t \in (-\delta_0, \delta_0)$ .

We now prove that the holomorphic vector field  $H^{a, \alpha}$  is tangent to the hypersurface  $M$ . Indeed, by a calculation we get

$$\begin{aligned} \rho_{z_1}(z_1, z_2) &= \frac{1}{2} + \frac{f_t(z_2, \operatorname{Im} z_1)}{2i}, \\ \rho_{z_2}(z_1, z_2) &= P_{z_2}(z_2) + f_{z_2}(z_2, \operatorname{Im} z_1). \end{aligned}$$

We divide the proof into two cases.

**a)  $\alpha = 0$ .** In this case,  $f(z_2, t) = Q_0(z_2)t$  for all  $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ . Therefore, by (i) and (ii) one obtains that

$$\begin{aligned} \operatorname{Re} H^{a, \alpha}(\rho(z_1, z_2)) &= \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) z_1 a(z_2) + \left( P_{1z_2}(z_2) + (\operatorname{Im} z_1) Q_{0z_2}(z_2) \right) iz_2 \right] \\ &= \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) \left( i(\operatorname{Im} z_1) - P_1(z_2) - (\operatorname{Im} z_1) Q_0(z_2) \right) a(z_2) \right. \\ &\quad \left. + \left( P_{1z_2}(z_2) + (\operatorname{Im} z_1) Q_{0z_2}(z_2) \right) iz_2 \right] \\ &= \operatorname{Re} \left[ iz_2 P_{1z_2}(z_2) - \left( \frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) a(z_2) P_1(z_2) \right] \\ &\quad + (\operatorname{Im} z_1) \operatorname{Re} \left[ iz_2 Q_{0z_2}(z_2) + \frac{1}{2} (1 + Q_0(z_2)^2) ia(z_2) \right] = 0 \end{aligned}$$

for every  $(z_1, z_2) \in M$ , which proves the theorem for  $\alpha = 0$ .

b)  $\alpha \neq 0$ . It follows from (iii), (iv), and (v) that

$$\begin{aligned}
& \operatorname{Re} H^{a,\alpha}(\rho(z_1, z_2)) \\
&= \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{f_t(z_2, \operatorname{Im} z_1)}{2i} \right) L(z_1) a(z_2) + \left( P_{z_2}(z_2) + f_{z_2}(z_2, \operatorname{Im} z_1) \right) i z_2 \right] \\
&= \operatorname{Re} \left[ \left( \frac{1}{2} + \frac{f_t(z_2, \operatorname{Im} z_1)}{2i} \right) \frac{1}{\alpha} \left( \exp \left( \alpha (i \operatorname{Im} z_1 - P(z_2) - f(z_2, \operatorname{Im} z_1)) \right) - 1 \right) a(z_2) \right. \\
&\quad \left. + \left( P_{z_2}(z_2) + f_{z_2}(z_2, \operatorname{Im} z_1) \right) i z_2 \right] \\
&= \operatorname{Re} \left[ \frac{1}{\alpha} \frac{i + f_t(z_2, \operatorname{Im} z_1)}{2i} \exp \left( \alpha (i \operatorname{Im} z_1 - f(z_2, \operatorname{Im} z_1)) \right) \exp(-\alpha P(z_2)) a(z_2) \right. \\
&\quad \left. - \frac{1}{\alpha} \left( \frac{1}{2} + \frac{f_t(z_2, \operatorname{Im} z_1)}{2i} \right) a(z_2) + \left( P_{z_2}(z_2) + f_{z_2}(z_2, \operatorname{Im} z_1) \right) i z_2 \right] \\
&= \operatorname{Re} \left[ \frac{1}{\alpha} \frac{i + Q_0(z_2)}{2i} \exp(-\alpha P(z_2)) a(z_2) - \frac{1}{\alpha} \left( \frac{1}{2} + \frac{f_t(z_2, \operatorname{Im} z_1)}{2i} \right) a(z_2) \right. \\
&\quad \left. + \left( P_{z_2}(z_2) + f_{z_2}(z_2, \operatorname{Im} z_1) \right) i z_2 \right] \\
&= \operatorname{Re} \left[ i z_2 P_{z_2}(z_2) + \left( \frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) \frac{\exp(-\alpha P(z_2)) - 1}{\alpha} a(z_2) \right] \\
&\quad + \operatorname{Re} \left[ i z_2 f_{z_2}(z_2, \operatorname{Im} z_1) + \frac{1}{2\alpha} \left( f_t(z_2, \operatorname{Im} z_1) - Q_0(z_2) \right) i a(z_2) \right] = 0
\end{aligned}$$

for every  $(z_1, z_2) \in M$ , which ends the proof.  $\square$

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